Decentralized Inventory Control for Large-scale Supply Chains

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Abstract—In this work, we illustrate a new decentralized optimization method for inventory control across large-scale supply-chains. The decentralized optimization method and the coordination algorithm provides a convenient approach to find the cooperative solution that is best for the whole supply-chain. Specifically, the scheme does not require each company to have explicit knowledge about the individual cost functions and the local constraints of the other companies. This factor helps eliminate the problems that result from informational deficiencies in large-scale supply-chains.

I. INTRODUCTION

Supply-chains are characterized by complex movement of goods, information, and funds between different levels of production and operation across companies and organizations. Typical networks show a geographically distributed structure that consists of manufacturers, distributors, and retailers. Operation decisions (such as shipment of parts or inventory level updates) require logistic coordination across this nonhomogeneous structure. Making such operational decisions is a very challenging problem when the actual decision (action) centers are distributed. Traditionally, contractual agreements and complex accounting schemes are used to ensure that the supply-chain works effectively during daily operations. However, these centralized solutions result in efficiency losses in a competitive dynamic market environment. In this work, we apply a decentralized optimization method [12] for inventory control across large-scale supply-chains.

Using the NASA space shuttle program supply-chain structure as an example, one can identify specific cases which highlight the inability of a centralized mechanism to handle the distributed information at each subcontractor level. NASA employs one of the most complex supply-chain structures for the development and the operations phase of the space shuttle. Reports by the Congressional Committee [16] and the Competitive Sourcing Task Force [7] identified major problems in the supply-chain structure, which resulted in delays and financial losses. Independently, they tie these problems to information and coordination deficiencies between the complex web of management centers, main contractors, and subcontractors. In such cases, decentralized coordination can provide an alternative mechanism that can correctly incorporate the distributed nature of the information and the decision-makers.

A. Case: NASA Orbiter Development Supply-Chain

One of the major problems that was seen during the development phase of the major subsystems was a significant overrun in costs. In the orbiter development, this problem is attributed [16] to the informational deficiencies that occurred in production planning between management, the lead center, the main contractor and the subcontractor.

In one specific case [16], delays in procurement of parts by a subcontractor were caused by the centralized accounting and financial reporting system, which did not allow the purchase to be made in order to meet current budget objectives. This resulted in an insufficient level of inventory for the following year’s manufacturing cycle as the subcontractor was unable to procure the parts at low cost because of the competition from other major aerospace companies. As a result, there was a domino effect of significant cost increase throughout the whole supply-chain. This situation demonstrates a key weakness of a centralized budgeting and financial planning system, as a central planner could not possibly account for all sub-contractor inventory levels and pricing strategies in making high level budgeting decisions.

In the optimization framework, the efficient operation of such a supply chain can be achieved by minimizing a global cost function, which is represented as a summation of all the operating costs from each of the manufacturing facilities. By minimizing this global cost, the optimal manufacturing and inventory levels of each facility, and the required amounts of goods that has to be transferred between these facilities, can be found. However, as illustrated in the space shuttle example, it becomes infeasible for a centralized unit to have or correctly model all the operational costs and manufacturing details of each company within such a large supply-chain. Thus, methods which allow decentralization of
this global optimization problem through local facility-based optimizations are crucial in obtaining efficient operations throughout the supply-chain.

B. Large-scale Supply-chain Model Characteristics and Decentralized Solution Approach

A key feature of this global optimization problem is that the optimization variables are distributed to each of the facilities. These variables essentially correspond to the local decision variables of the companies that form the supply-chain. For example, in the space shuttle example, the number of parts that the subcontractor purchases is a local decision variable of the subcontractor. However, this decision affects the whole supply-chain, as these parts are integrated and shipped to the contractor facilities where they are used in the production of a space shuttle subsystem. Although such a characteristic is a major complicating factor for a centralized optimization, it provides a natural property that can be exploited to obtain a decentralized solution method. Specifically, this characteristic allows the global optimization problem to be mathematically decomposed into a series of local optimization problems, in which each facility minimizes its local operating cost to find its local manufacturing level subject to local constraints that correspond to the internal operations of the facility and the global operational constraints which interconnect it to other facilities. The global operational constraints ensure that the flow of materials or finished goods from one facility to another is mathematically consistent — products shipped by contractor A to contractor B is the amount of products received by contractor B from contractor A. Without such global operation constraints, the local optimal manufacturing level of each company and the necessary shipment decisions to and from its facility does not match the local optimal decisions of the other companies that it is operationally tied to within the supply-chain.

For large-scale systems of this kind, decentralized coordination provide a solution in which different levels of the network generate joint decisions by agreement on their local operational constraints. To solve this problem, consider the decentralized optimization method in which the local operating cost function of each facility is augmented with a penalty function that results in this facility incurring costs if its global operational constraints are not satisfied. As a result, the new cost function of each facility is a weighted sum of the penalty function and its local operating cost, which is scaled by a penalty parameter. This method can be implemented in a coordination scheme in which each company optimizes its penalty-augmented cost function in a sequential round-robin structure and transmits its manufacturing decisions to the companies with which it has joint operations. In the third section it will be shown that if the companies keep their penalty parameter ratios constant and progressively decrease their individual penalty parameters to zero, the coordination algorithm will result in each decision-maker converging to manufacturing decisions which will satisfy the global operational constraints. In addition, the manufacturing decision of each company will correspond to the global optimal answer, which is the solution of the global optimization problem.

C. Related Work : Economics literature, game theoretic solutions, multi-agent systems and distributed optimization methods

Both economics and game theory literature contains a wide range of approaches pertinent to analyzing and classifying equilibria among a large number of parties [14], and providing strategies to obtain desirable outcomes for each decision-maker [8], [18].

Specifically from the optimization framework, equilibrium programming in economics provides a coherent framework [13] which takes into account the existence of multiple decision-makers. However, decentralized computation of such optimization problems has not been the core research area because the main interest is in finding these equilibria, rather than solving the optimization problem in a decentralized fashion. In order to model the interactive nature of the decision-makers, market based algorithms have been developed such as in [4], [20], [21]; the decision-makers (agents) and the supervisory controller (markets auctioneer) exchange offers (supply-demand, bid-market price of the commodity) in an effort to find the optimal solution through a competitive equilibria [22] convergence process.

In the distributed optimization literature, numerous solution techniques and algorithms have been presented to solve the decomposed optimization problems [3], [6]. To optimize multiple objective functions at the same time, multi-objective optimization methods [15], [10], [5] have been extensively developed. Specifically, distributed computation of Pareto-optimal solutions has been studied in [19], [9]. However, in these multi-party negotiations a “neutral mediator” acts as an upper level iteration above subsystem optimizations. Thus, the structure of the problem doesn’t allow decentralization based on operational constraints, but rather adapts a hierarchical structure. This is a structure that needs to be eliminated for large-scale supply-chains to incorporate the distributed nature of the information and the mathematical models.

In the next section, we illustrate the fundamental idea behind the decentralized optimization through a simple inventory control problem for a distributed supply-chain. This problem will later be solved in a decentralized fashion to illustrate our method.

II. SUPPLY-CHAIN MODEL AND PROBLEM

Current development programs in the space launch industry are rapidly changing to meet the increased demand in low-cost access for commercial payloads to space [1]. The trend is towards smaller programs in which private companies demonstrate and integrate new technologies in fields such as propulsion systems and conceptual design. The manufacturing of different hardware elements of the launch vehicles (such as the engines or the guidance and control computers) are distributed to multiple “responsibility centers”. This concept is illustrated in Fig. 1. Coupled with the highly specialized nature of the aerospace industry, this results in manufacturing supply-chains being governed by independent partners with incomplete knowledge about the
whole structure. In such a case, for the supply-chain to work efficiently (i.e., optimally), each partner within the supply-chain should naturally cooperate with the centers that it is working with. Furthermore, a key requirement for such a system to work efficiently is to dynamically anticipate the random launch market and act accordingly while the coordination and cooperative decision-making is done with limited information.

Consider the hardware development and manufacturing supply-chain shown in Fig. 2. This model consists of a series of companies (“Company Z”, “Company Y”) that manufacture the controller boxes for the expendable stage of the rapid launch space vehicles used by the “Launch Corporation W”. In this example, the volatile launch market that can only be predicted for the next three months period. At the beginning of each manufacturing cycle, the supply-chain coordination problem is to find the optimal level of manufacturing (and thus inventory) of the control boxes for the next quarter.

The mathematical models of the companies involved in the supply-chain and the specifics of the supply-chain coordination problem is given below.

**Company Z** is a hardware designer and integrator, and works as a subcontractor to firms which develop commercial-off-the-shelf GN&C solutions. It constructs avionics and controller boxes; integrating processors, inertial measurement units, GPS receivers, or sets of sensor suites bought from the original equipment manufacturers (OEMs).

**Company Y**’s specialty is in producing custom guidance, navigation and control software solutions for the technology demonstrator firms. Its work involves embedding their guidance and navigation algorithms in custom avionics boxes ordered from companies such as Company Z. The end product is a complete commercial-off-the-shelf GN&C hardware and software product used at the guidance and navigation subsystems of air and space vehicles.

In this supply-chain (Fig. 2), Company Y has a “sole supplier” contractual agreement to provide customized controller hardware and software to be used in Launch Corporation W’s fleet of space launch vehicles. Specifically, the custom controller unit is used in the expendable stage of the launch vehicles. Company Y also has a main contractor-subcontractor agreement with Company Z. In this agreement, Company Z supplies the essential hardware which consists of the controller processors and the attached sensor units. Depending on the launch rates and the schedules, “Launch Corporation W” orders these customized boxes to be used for combustion and separation control in their launch vehicles’ expendable stages. At every financial quarter, Launch Corporation W produces a probabilistic launch market model for the next quarter. In this report, the number of launches is modeled with a random positive number \( d \) and a corresponding probability density function \( \mu(d) \). Based on such market models, the subcontractor and the main contractor (Company Y and Z respectively) aim to minimize the supply-chain’s expected operating costs. They do this by finding the optimal manufacturing amount of the control boxes for the next quarter.

In the next section, the individual mathematical models for the subcontractor and the main contractor are illustrated. Using their local models, both of the parties can calculate the optimal level of manufacturing for the supply-chain from their own perspective. It turns out that the solutions found are not only different than each other, but also not optimal for a commonly used collective performance metric. Later, a decentralized optimization method stemming from the original results in [12] is used to solve this coordination problem. The method relies on each decision-maker reaching an agreement in order to find the global optimal solution of the supply-chain.

### A. Subcontractor’s Perspective

As a part of its three month period operations, the subcontractor, Company Z, would like to keep an optimal level of production of controller units based on the random demands of the launch vehicle operations. Its operating costs are tied to the profits and losses associated with the sale of \( z \) units of controller hardware to Company Y.

For every unit sold, Company Z makes \( c_s \) dollars of profit. In the case in which the demand is less than the controller boxes shipped, they are returned back to its inventory. This results in total loss of profit and some internal restocking costs. For every unit returned, the inventory cost is \( h_s \) dollars of loss and \( h_s > c_s \). When the number of units shipped to the manufacturer is less than the current demand, the company operates extended hours meeting the demand but, there is a slight loss of profit compared to the direct up-front sales. For every unit sent as a second shift sale, \( p_s \) dollars of profit is made and \( 0 < p_s < c_s \).

Under this operation model, the subcontractor’s problem is to find the optimal number of controller boxes, \( z^0 \), to

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1 In this model, the number of launches is approximated with a continuous random variable rather than a discrete random variable for ease of computation. For example, a typical launch market model can be a continuous exponential random distribution \( \mu(d) = \frac{d}{\lambda} e^{-\frac{d}{\lambda}} \), with mean \( \lambda \) and variance \( \lambda^2 \). Here, to find the probability that the number of launches is going to be less than or equal to \( a \) (i.e. \( P(d \leq a) \)), one can integrate the probability density function \( \mu(d) \) over the interval \( [0, a] \) to obtain: \( P(a) = 1 - e^{-\frac{a}{\lambda}} \). Here, \( \Phi \) is the distribution function of the number of launches.
be manufactured and then shipped to the main contractor based on the launch vehicle demands for the next quarter. Under a demand forecast \(\mu(d)\), the optimal operation of the subcontractor can be compactly described as minimizing the total expected operation cost:

\[
C(z) := \mathbb{E}_{d \geq 0} \{-c_s z - p_s \max(0, d - z) + h_s \max(0, z - d)\}
\]  

(1)

Notice that, the operation cost is a combination of

- profits from direct sale \(-c_s z\),
- profits from extra units to be produced for unmet demand sale \(-p_s \max(0, d - z)\), and
- losses from returned controller units \(h_s \max(0, z - d)\).

In a compact structure, the subcontractor’s optimal operation can be described via the following minimization.

\[
\min_{z \geq 0} C(z)
\]  

(2)

In this form, Company Z’s problem is a simple one step inventory control problem. It is shown in [11] that the unique solution \(z^o\), that minimizes \(C(z)\), satisfies the implicit function

\[
\Phi(z^o) = \frac{c_s - p_s}{h_s - p_s}
\]  

(3)

where \(\Phi\) corresponds to the integral of the probability density function \(\mu\) of the vehicle launches:

\[
\Phi(a) := \int_0^a \mu(d) \delta d
\]  

(4)

For example, for any given number \(a\), \(\Phi(a)\) corresponds to the probability that the number of launches is going to be less than or equal to \(a\) (i.e. \(P(d \leq a)\)). Notice that \(\Phi\), which is called the distribution function, will always take values between \([0, 1]\).

As shown in (3), the optimal solution \(z^o\) has a distribution function value \(\frac{c_s - p_s}{h_s - p_s}\). Thus, if the above integral can be calculated in closed form, then (3) can be either solved iteratively or analytically to find \(z^o\) for any given random launch distribution. For example, let’s assume that the industry random launch model can be described by a simple exponential random distribution \(\mu(d) = \frac{1}{d^o} e^{-\frac{d}{d^o}}\) where \(d^o\) is the expected number of launches (the mean of distribution \(\mu(d)\)). If Company Z’s operation parameters are \(p_s = 0.8c_s\), \(h_s = 1.2c_s\) then the optimal number of controller units to be manufactured satisfies \(1 - e^{-\frac{c_s}{h_s}} = 0.5\) through (3). In this case, \(z^o = 0.693d^o\).

If the expected number of launches for the next quarter, \(d^o\), corresponds to 100 then the subcontractor projects that it should build and sell approximately 70 units to the main contractor. The main contractor then would embed its software and algorithms in those units. If in the next quarter, more than 70 launches occur, then the company expects to do second-shift sales to meet the extra demand. If, in the next quarter, there are fewer than 70 launches, then the extra units are returned to its inventory. The subcontractor’s solution reflects a conservative outlook where it is more inclined towards doing second shift sales with diminished profits rather than overspending the demand and facing the higher restocking fees for unsold units. However, the main contractor has a different operation principle as explored in the next section.

B. Main Contractor’s Perspective

The main contractor would like to minimize its own operations costs which have a different structure than those of the subcontractor. Company Y pays \(c_b\) dollars for each controller unit that it buys and makes \(c_m\) dollars profit from each of the units sold with its software installed. If the launch demand is lower than the \(m\) controller units bought, it is returned to the subcontractor and the whole purchase price, \(c_b\) dollars per unit, is collected back. However, there is a loss coming from the local inventory keeping charges, thus only \(h_m\) dollars can be effectively earned back for each unit sent: \(h_m < c_b\). If the launch demand is higher than the number of units bought, the main contractor makes second shift purchases from the subcontractor. However, there are increased production costs because of extra hours of operation. As a result, the purchase price, \(p_m\) dollars per unit, becomes higher: \(p_m > c_b\).

With this, the contractor’s problem is to find \(y^o\), the optimal number of controller unit purchases, that would minimize its operation costs:

\[
D(y) := \mathbb{E}_{d \geq 0} \{c_b y - c_m d + p_m \max(0, d - y) - h_m \max(0, y - d)\}
\]  

(5)

\(D(y)\) represents a combination of

- profit from sale to the Launch Corporation \(W\) : \(-c_m d,\)
- initial purchase price of controller units from the subcontractor : \(c_b y,\)
- emergency purchase to meet unmet demand: \(p_m \max(0, d - y)\), and
- the diminished return for sending back the excess units to the subcontractor: \(-h_m \max(0, y - d)\).

The main contractor’s optimal operation is tied to solving the following problem:

\[
\min_{y \geq 0} D(y)
\]  

(6)

Just like the subcontractor’s case, the optimal solution \(y^o\) satisfies [11] the implicit cumulative distribution function:

\[
\Phi(y^o) = \frac{c_b - p_m}{h_m - p_m}
\]  

(7)

For the same exponential distribution of launch numbers, and local operation parameters \(p_m = 2c_b\) and \(h_m = 0.5c_b\), the optimal solution of controller units to be purchased is \(y^o = 1.0986d^o\). From the main contractor’s perspective, for the supply-chain to operate optimally, it should buy \(y^o\) (which is roughly 110% of the expected number of launch operations) controller units from Company Z. The main contractor’s optimal choice of purchase reflects the nature of its operation parameters. Notice that it incurs a hefty cost \((p_m = 2c_b)\) for secondary purchases when the initial demand isn’t met. Thus, the main contractor is inclined towards overshooting the demand rather than doing expensive second shift sales.
The different solutions found by the main contractor and the subcontractor reflect their respective outlook based on their operational characteristics. In this case, for the supply-chain to work efficiently (i.e. optimally), the main contractor and the subcontractor should naturally cooperate to find a joint optimal solution. A commonly used cooperation metric [2] is the minimization of the total operation costs (which is equal to the maximization of the profits) of each business unit. A possible joint operation metric for this “global optimization problem” can be defined as the minimization of the total operation costs (which is different from the projected solutions by Company Z) calculated as the following equation:

\[
\min_{y,z \in \mathbb{R}} [C(z), D(y)]
\]

subject to

\[
\begin{align*}
& y \geq 0 \\
& z \geq 0 \\
& y = z
\end{align*}
\]

Assuming the volatile launch market described by the same exponential random distribution, the operating costs of the companies can be explicitly written as follows:

\[
C(z) = -h_s d^z + (h_s - c_s)z + (h_s - p_s) d^z e^{-\frac{z}{\mu}}
\]

\[
D(y) = (h_m - c_m) d^y + (c_b - h_m) y + (p_m - h_m) d^y e^{-\frac{y}{\mu}}
\]

In this case, because of a lack of a centralized decision maker, the subcontractor and the main contractor should coordinate to find the global optimal manufacturing level. Explicitly, the centralized optimization problem corresponds to simultaneously minimizing the operating costs of the subcontractor \((C(z))\) and the main contractor \((D(y))\), while making sure that they are in agreement with each other \((y = z)\). This specific problem can be formulated as a multi-objective optimization:

In this section, the solution of this problem will be shown to be a function of the operational parameters of both the subcontractor \((c_s, p_s, h_s, p_s)\) and the main contractor \((c_m, p_m, h_m)\). In general, a central computational facility is needed to solve this problem because of the inherently coupled nature of the solution. Later, the simplified decentralized optimization method is utilized to find this global optimal solution without the need for explicit knowledge about the mathematical models (i.e. the operational parameters) of the other party.

C. Global Optimization Problem

The global optimization problem for the whole supply-chain in our example can be described by the minimization of the total operating costs \((\frac{C(m)}{c_s m} + \frac{D(m)}{c_m})\). This is given in the following equation:

\[
\min_{m \geq 0} \frac{1}{c_s} C(m) + \frac{1}{c_m} D(m)
\]

It is shown in [11] that the unique solution \(m^o\) that minimizes the above equation, satisfies the implicit function

\[
\Phi(m^o) = \frac{(p_m - c_b) + \frac{c_s}{c_b}(c_s - p_s)}{c_s} (c_s - p_s)
\]

For example, if the subcontractor’s profit margin is one tenth of the sale price (i.e. \(c_b = 10c_s\)) and the profit of the main contractor per unit sale of the unit is twenty times more than the profit of the subcontractor (i.e. \(c_m = 20c_s\)) then, the global optimal solution \((m^o)\) can be written in the implicit form:

\[
\Phi(m^o) = \frac{10c_s + 4c_b}{8c_s + 15c_b}
\]

Assuming the exponential random distribution as the launch model, \(\mu(d) = \frac{1}{\mu_d} e^{-\frac{d}{\mu}}\), the global optimal is calculated as \(m^o = 0.9383d^o\). Note that this global solution is different from the projected solutions by Company Z \((z^o = 0.693d^o)\) and Company Y \((y^o = 1.0986d^o)\). Neither of the companies could correctly estimate the global optimal solution by just using the local information that embeds only their own operational characteristics.

\[
\begin{align*}
\Phi^{\text{model}}(z^o) &= \frac{10c_s + 4c_b}{8c_s + 15c_b} \\
\Phi^{\text{model}}(y^o) &= \frac{10c_s + 4c_b}{8c_s + 15c_b}
\end{align*}
\]

The set of solutions for this multi-objective optimization problem is shown in Fig. 3. Each point in this figure corresponds to an agreement \(y = z = m\) and maps the operating costs that come with this agreement to the respective contractors. Specifically, the portion represented with circles corresponds to the Pareto optimal solutions of this optimization problem. This portion is known in the literature[15] as the Pareto optimal frontier. For this problem, the Pareto optimal frontier corresponds to agreements...
that take values between the local optimal solutions of the companies: $z^0 ≤ m ≤ y^0$. Notice that any solution in this frontier is Pareto optimal and can be claimed as a valid solution to the multi-objective optimization problem. This is illustrated in Fig. 3.

The Pareto optimal solutions of the centralized optimization problem represent only a section of equilibria for such systems. In Fig. 3, the remainder of the agreements that are outside of the Pareto optimal frontier fall into a bigger class of solutions called the Nash equilibria. As illustrated in Fig. 3, such Nash equilibria are not Pareto optimal because there are alternative agreements which can simultaneously increase the profits of both parties. For extended theoretical classifications, refer to [12]. Many large-scale systems with multiple levels have actually been designed to operate with solutions that correspond to Nash equilibria [17]. The main aim of such work has been to decompose the large-scale systems into a multiple-level structure in which the optimization for each level is simplified considerably and the individual optimizations can be carried out recursively. However, this simplification is achieved at the expense of loss of global efficiency because, such a structure results in operation of the large-scale system at the Nash equilibria.

D. Decentralized Optimization

Using the baseline example, if the subcontractor and main contractor start to cooperate, they will initially face a conflict. This is because in mathematical terms, the local solutions ($y$ and $z$) are infeasible in the sense that $y > z$; the amount of controller units that the subcontractor would like to sell, $z$, is less than the amount of units that the main contractor would like to buy, $y$.

To solve this problem, consider a simple sequential iterative scheme where the subcontractor and main contractor exchange offers and counteroffers. The major challenge in this is that both the subcontractor and main contractor have knowledge only about their own operations described by their cost functions. The main aim of the scheme would be to reach an agreement ($y = z$) which will be globally optimal ($y = z = m^0$), while the independent decision-makers make self-interested decisions with incomplete knowledge about the other party’s cost function. To aid them in reaching an agreement, both of the parties can embed the only common knowledge that they have, the agreement constraint $y = z$, as a penalty function of quadratic form $(y - z)^2$ to their local costs.

To penalize their violation of agreement, each of the parties introduce a local penalty parameter $β_s > 0$ and $β_m > 0$ respectively. These parameters specify the importance of constraint violation in the subcontractor’s and main contractor’s problems. However, instead of directly penalizing the constraint violation, they penalize their own local operation cost functions. In this structure, Company Z’s and Company Y’s local optimization problems look like the following:

**Subcontractor’s Optimization**

$$\min_{z \geq 0} \beta_s[-c_s z + L_s(z)] + (y - z)^2$$  \hspace{1cm} (15)

**Contractor’s Optimization**

$$\min_{y \geq 0} \beta_m[c_b y - c_m d_o + L_m(y)] + (y - z)^2$$  \hspace{1cm} (16)

If the main contractor makes an offer $\bar{y}$, the subcontractor can select an arbitrary $β_s$ and can solve (15) to come up with the optimal amount of controller units that it is willing to ship; $z^*$, which can be written as:

$$\Phi(z^*|\bar{y}, β_s) = \frac{c_s - p_s}{h_s - p_s} \frac{2(\bar{y} - z^*)}{\beta_s(p_s - h_s)}$$  \hspace{1cm} (17)

Notice that $z^*$ has functional dependence on the offer $\bar{y}$ and the subcontractor’s local choice of penalty parameter, $β_s$. Thus the solution is represented as $(z^*|\bar{y}, β_s)$, highlighting this functional dependence. $z^*$ is counter-offered to the main contractor as the number of units that the subcontractor is willing to ship, given $\bar{y}$.

The main contractor denotes the offer $(z^*)$ received from the subcontractor as $\bar{z}$. From the main contractor’s perspective, the optimal amount of controller boxes to be shipped for a proposed $\bar{z}$ is $(y^*|\bar{z}, β_m)$:

$$\Phi(y^*|\bar{z}, β_m) = \frac{p_m - c_b}{p_m - h_m} \frac{2(y^* - \bar{z})}{β_m(p_m - h_m)}$$  \hspace{1cm} (18)

Given the offer of $\bar{z}$, the main contractor is willing to buy $y^*$ controller units. Looking at this process from the main contractor’s point of view, the size of error or disagreement at any given time can be written as $(y - z) = \frac{β_m(p_m - h_m)}{2}[(p_m - h_m)\Phi(z|y) - \Phi(z^*)]$. As $\Phi(z|y), \Phi(z^*)$ are distribution functions, they are bounded between $[0, 1]$. Using this, the size of disagreement can be approximated via the following equation:

$$e = |y - z|$$  \hspace{1cm} (19)

$$≤ β_m(p_m - h_m)$$  \hspace{1cm} (20)

The same case holds true from the subcontractor’s point of view:

$$e = |y - z|$$  \hspace{1cm} (21)

$$≤ β_s(h_s - p_s)$$  \hspace{1cm} (22)
As $\beta_s \to 0$ and $\beta_m \to 0$, the companies reach a perfect agreement (i.e. $e \to 0$) and the convergence rate is a linear function of $\beta_m$ and $\beta_s$. So, the simplest solution to this problem would be if the companies gradually decrease their $\beta_s$ and $\beta_m$ selection to zero in the limit. Then, the system will converge to a solution in which $y^* = z^*$. This entire process is illustrated in Fig. 4.

Given this insight, let’s consider the solution in the limit case: ($y^* = z^*$). Defining $\alpha = \frac{\beta_s}{\beta_m}$, then the solution can be compactly written as

$$\Phi(z^* = y^*) = \left(\frac{p_m - c_b}{\alpha(h_s - p_s)}\right) + \frac{(p_m - h_m)}{\alpha(h_s - p_s)} \alpha$$

(23)

If the decision-makers ensure that $\alpha$ is equal to $\frac{c_m}{c_s}$, the limit solution would correspond to the global optimal answer found in the previous section. A standard approach to achieve this is a two step procedure in which each decision-maker scales its cost function by its unit sale value and iteratively solves its local optimization based on the counteroffer received from the other party. Once this process converges to a solution, if the disagreement is more than some amount denoted as tolerance (i.e. $|y^* - z^*| > \text{tolerance}$), then the decision-makers choose a common scaling factor to decrease their penalty parameters by it. This ensures that the subcontractor and main contractor will still keep the $\alpha$ ratios. This two step procedure is repeated until the disagreement is within the tolerance amount.

The numerical solution of the decentralized optimization is illustrated in Fig. 5. After eight hundred iterations, the subcontractor and main contractor reach an agreement to produce $y^* = z^* = 0.9390a^0(\pm0.0001d^0)$ amount of controller units. This solution is within \%0.1 of the global optimal solution, which is $m^0 = 0.9383d^0$.

III. Generalization of Results

The results of convergence and optimality shown in the supply-chain is generalized [12] to a general class of problems wherein coordination is governed by multiple decision makers with limited centralized information. Below these results are summarized. For extended analysis and proofs of our technique, refer to [12].

A. Convergence

Sum of local cost functions and the penalty functions for each constraints provides a convenient global metric for which sequential subsystem optimizations lead to convergence. From the perspective of any arbitrary $i_{th}$ decision maker, the global cost metric can be broken into two pieces representing the $i_{th}$ subsystem’s local optimization function $F_i(x_i, \beta_i | \{x_j\}_j)$ and the remainder defined as its complement, $F_i(x \setminus x_i, \beta \setminus \beta_i)$ in which $x_i$ and $\beta_i$ is the $i_{th}$ decision maker’s optimization and penalty parameter. In addition $P$ is a penalty function:

$$F(x, \beta) = F_i(x_i, \beta_i | x \setminus x_i, \beta \setminus \beta_i) + F_i(x \setminus x_i, \beta \setminus \beta_i)$$

Notice that $F$ is a function of $x_i$ and $\beta_i$ for given $x \setminus x_i$ (and $\beta \setminus \beta_i$). Thus, it is the same $x_i$ which minimizes $F_i(x_i, \beta_i | \{x_j\}_j)$ and $F(x_i, \beta_i | x \setminus x_i, \beta \setminus \beta_i)$. This shows that doing an optimization on $F(x_i, \beta_i | \{x_j\}_j)$ actually corresponds to doing an optimization on $F(x, \beta)$ while fixing $x \setminus x_i$ and $\beta \setminus \beta_i$. If this is recursively carried by each of the $m$ decision makers then it is indeed a nonlinear block optimization iteration on $F(x, \beta)$. As the penalty parameters are eventually decreased, the block iterations lead to convergence for this global cost metric.
B. Optimality

When converged, the first-order analysis of the local optimizations indicate[12] that the decentralized optimal solutions will have identical (or parallel) Lagrange multipliers for the interconnecting constraints. This specific characteristic is indeed necessary for the centralized optimality (i.e. Pareto optimality) of the decentralized solution. The solution of the sequential round-robin optimization is a feasible Nash equilibrium and corresponds to a decentralized optimal solution.

Further, the decentralized optimal solutions of the penalty based method will have these overlapping Lagrange multipliers, and thus satisfy the first order necessary conditions for Pareto optimality. The second-order analysis indicate that decentralized optimal solutions of non-convex problems will be locally Pareto optimal solutions if they have weak interconnections or strong local-convexity[12]. Fig. 6 summarizes how the decentralized solutions are mapped to centralized ones under these conditions.

IV. CONCLUSIONS

In this work, we have described a solution a standard inventory control problem using a decentralized optimization technique. For this problem, the approach provides an easily implementable solution to find the global optimal solution without the exchange of mathematical models or without the need for a centralized location to solve the problem. The method [12] is applicable to a general description of a large-scale dynamic system structured around independent decision-makers with incomplete models.

REFERENCES